Composable Core-sets for Determinant Maximization Problems via Spectral Spanners

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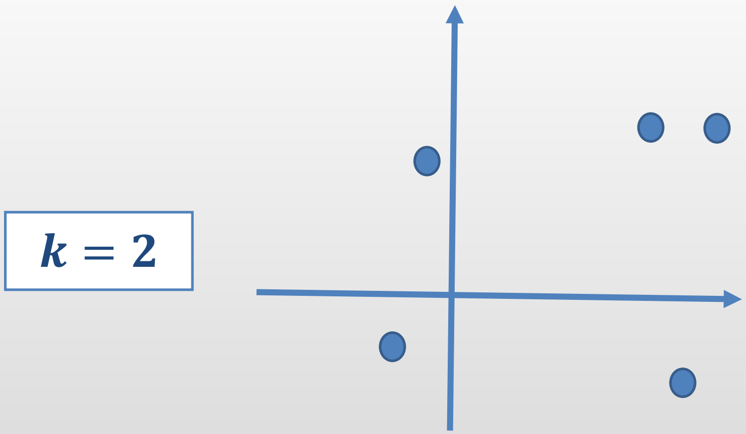
Volume (Determinant) Maximization Problem

Input: a set of n vectors $\mathrm{V} \in \mathbb{R}^d$ and a parameter $k \leq d$,



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- Parallelepiped spanned by the points in S

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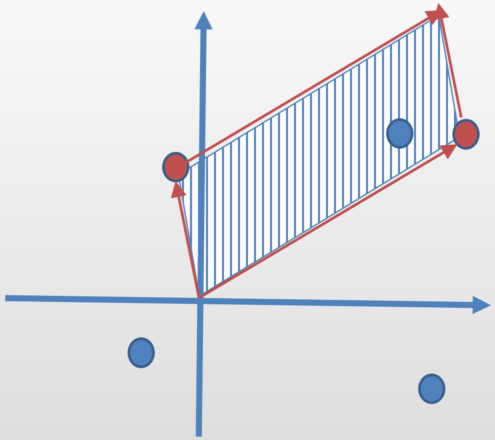

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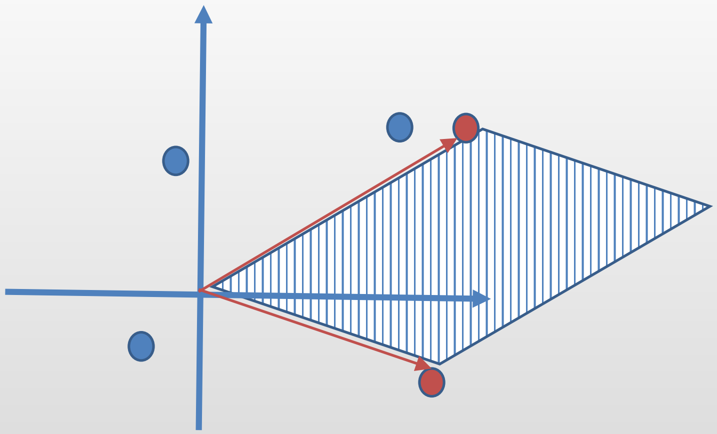

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$$\begin{bmatrix} v_1 \ v_2 \dots v_n \end{bmatrix}$$

$$\mathbb{V}$$

Equivalent Formulation:

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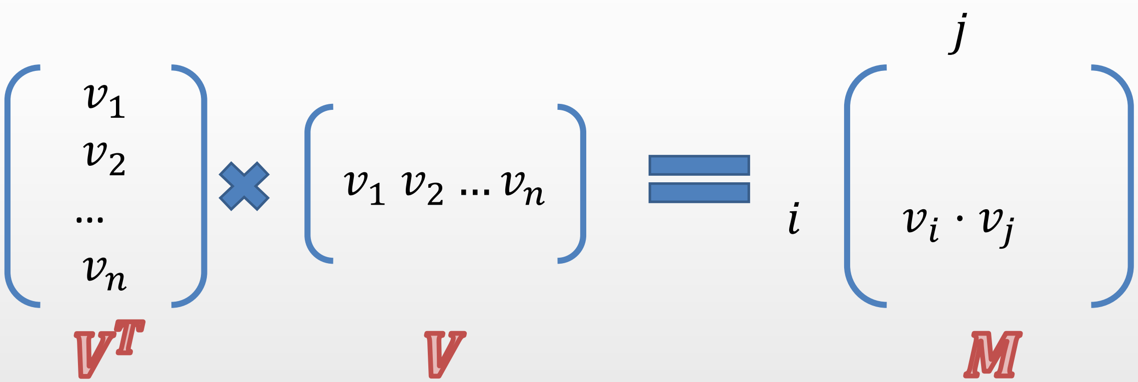

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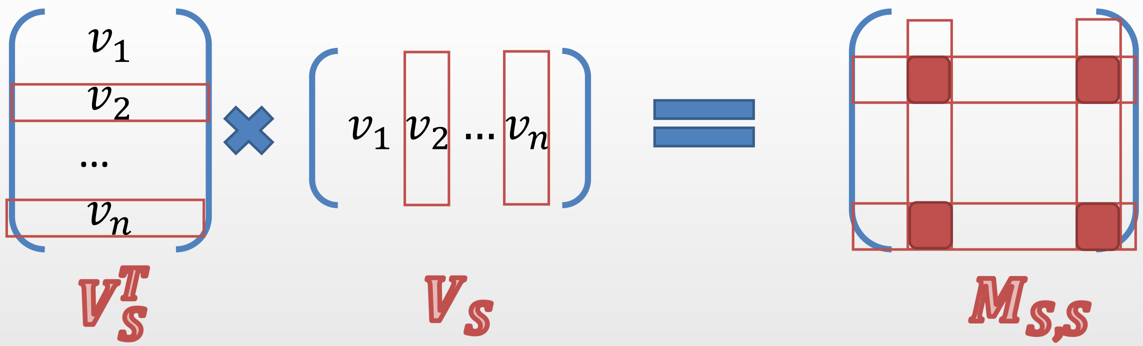

Equivalent Formulation:

Reuse V to denote the matrix where its columns are the vectors in V

- Let M be the gram matrix $V^T V$
- Choose S such that $\det(M_{S,S})$ is maximized

$M_{i,j} = v_i \cdot v_j$

$\det(M_{S,S}) = Vol(S)^2$

What is known?

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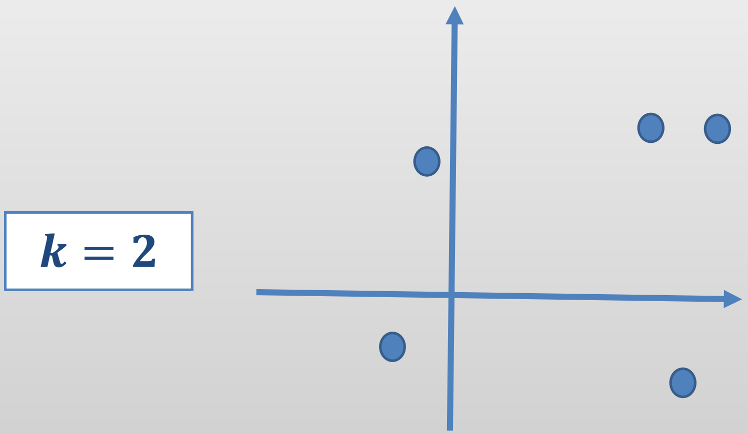

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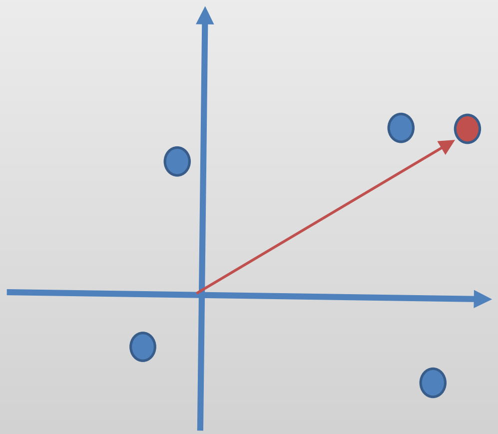

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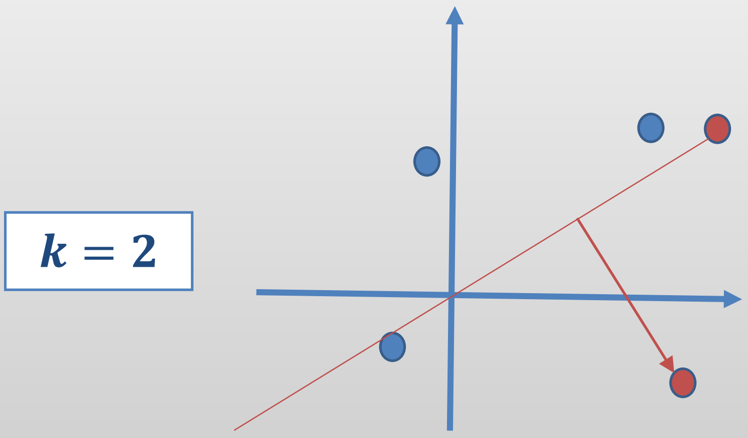

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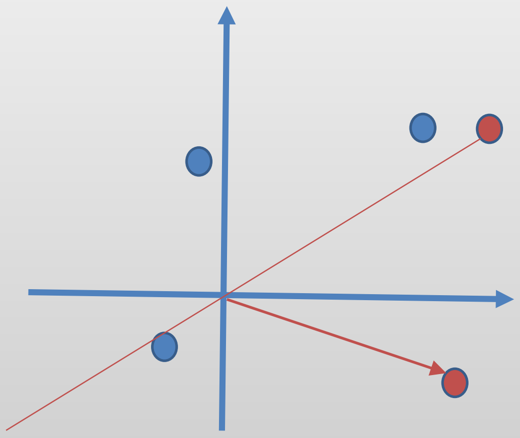

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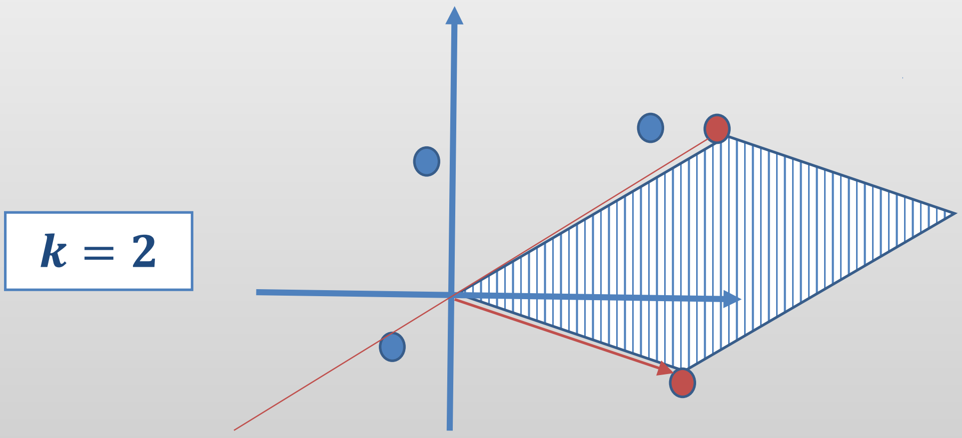

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- Greedy performs very well in practice

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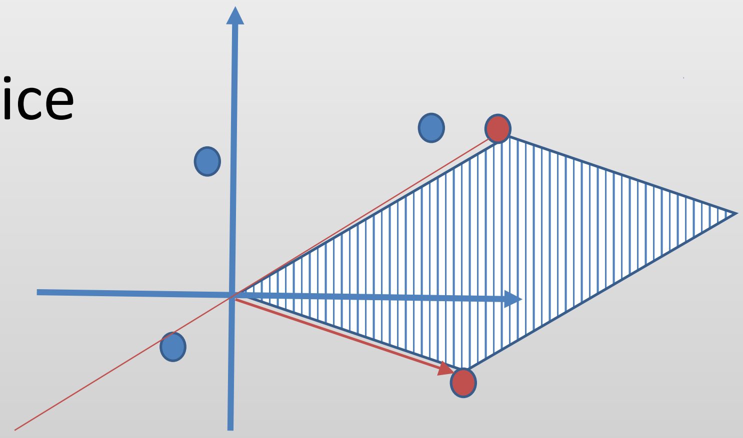

Determinantal Point Processes (DPP)

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- Lots of effort for solving the problem in massive data models of computation [MJK'17, WIB'14, PJG+'14, MKSK'13, MKBK'15, MZ'15, MZ'15, BENW'15]
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Composable Core-sets

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Core-sets [AHV’05]**:** a subset $\boldsymbol{U}$ of the data $\boldsymbol{V}$ that represents it well

Solving the problem over $\boldsymbol{U}$ gives a good approximation of solving the problem over $\boldsymbol{V}$

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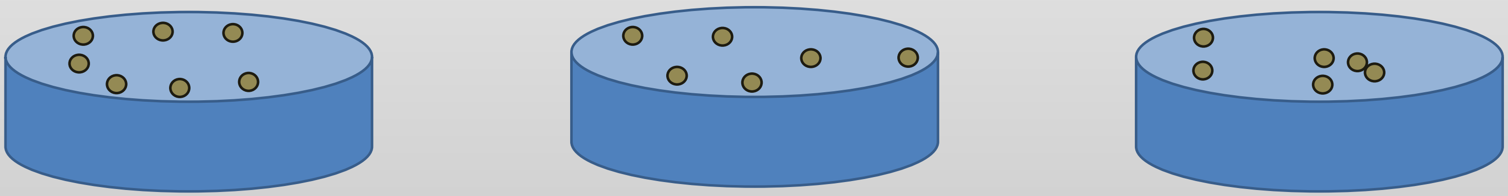

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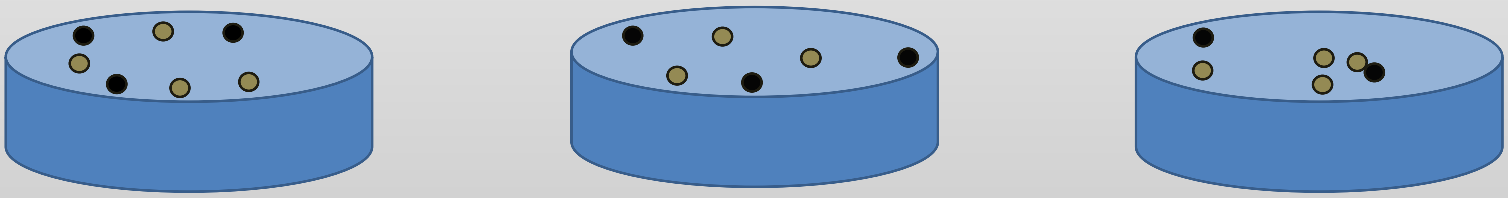

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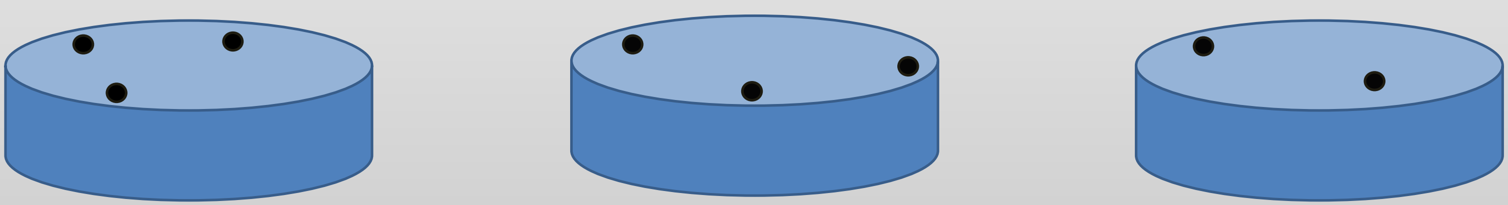

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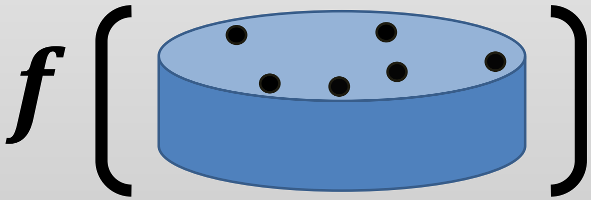

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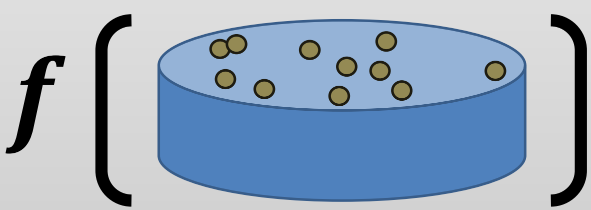

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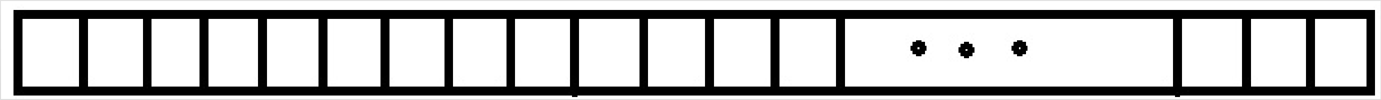
- ✓ Composable Core-sets have been studied for the **diversity Maximization** problems, for other notions of diversity: minimum pairwise distance, sum of pairwise distances, etc.
- ✓ Determinant maximization is a higher order notion of diversity

Applications: Streaming Computation

- **Streaming Computation:**
  - Processing sequence of n data elements "on the fly"
  - limited Storage



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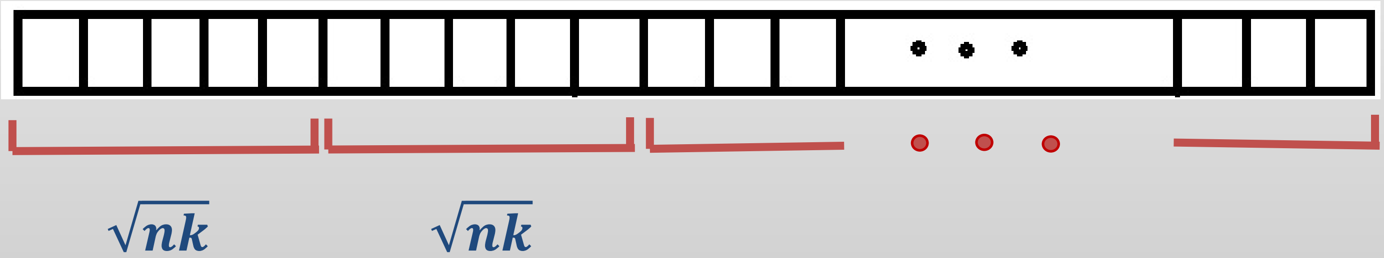

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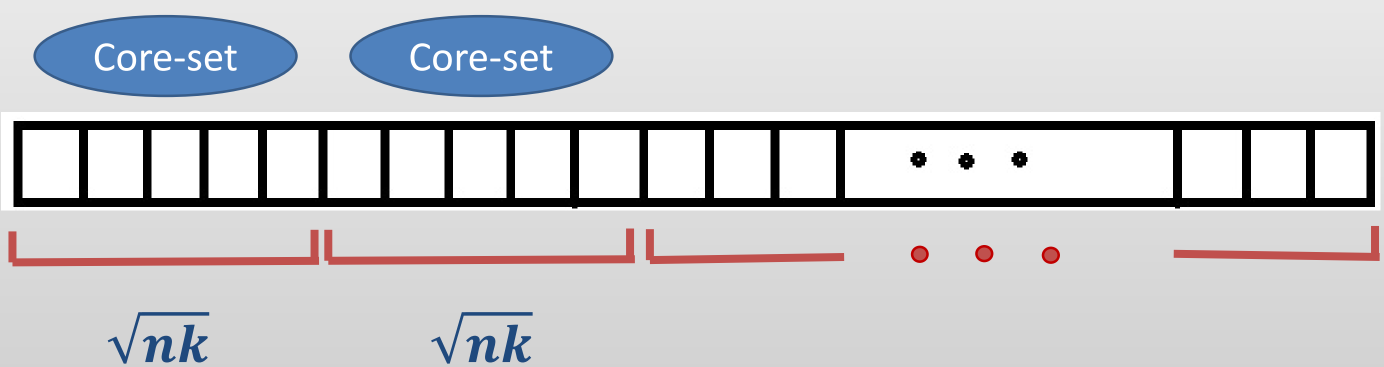
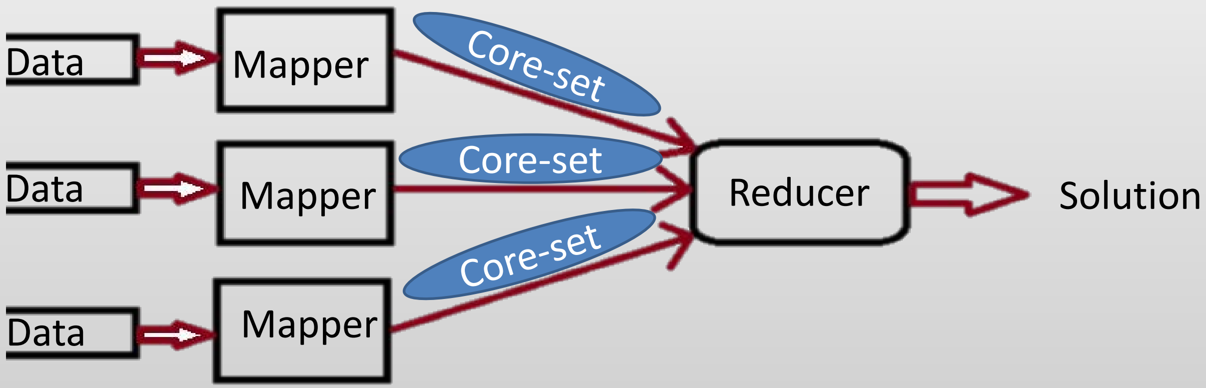
- **c-Composable Core-set of size k**
  - Chunks of size $\sqrt{nk}$, thus number of chunks = $\sqrt{n/k}$
  - Core-set for each chunk
  - Total Space: $k\sqrt{n/k} + \sqrt{nk} = O(\sqrt{nk})$
  - Approximation Factor: c



Applications: Distributed Computation

- Streaming Computation
- **Distributed System:**
  - Each machine holds a block of data.
  - A composable core-set is computed and sent to the server
- **Map-Reduce Model:**
  - One round of Map-Reduce
  - $\sqrt{n/k}$ mappers each getting $\sqrt{nk}$ points
  - Mapper computes a composable core-set of size k
  - Will be passed to a single reducer



Can we get a composable core-set of small size for the determinant maximization problem?

Results

Composable Core-sets for Determinant Maximization:

Algorithm:
There exists a polynomial time algorithm for computing an $\tilde{O}(k)^k$ -composable core-set of size $\tilde{O}(k)$ for the k-determinant maximization problem.

Lower bound:
Any composable core-set of size $k^{O(1)}$ for the k-determinant maximization problem must have an approximation factor of $\Omega(k)^{k(1-o(1))}$.

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➢ Note the gap with the approximation factor of the best offline algorithm: e^k

Overall Picture

Spectral Spanners

→

Generalizing the notion of

Combinatorial Spanners

Overall Picture

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Composable Core-sets for many spectral optimization problems

Streaming algorithm

Distributed algorithm

Map-Reduce algorithm

Overall Picture

Spectral Spanners

Easy case $\boldsymbol{k = d}$

Spectral Spanners ($\boldsymbol{k = d}$)

- Spanners: sparsifying a graph while preserving distances between nodes.

- Spectral Spanners: sparsifying a point set while preserving distances to hyperplanes.
  - similar to core-sets for width [AHV'05]

Spectral Spanners ($\boldsymbol{k = d}$)

Given a set of points V

A subset $\boldsymbol{U \subseteq V}$ is a **α-spectral spanner** of V if for every $\boldsymbol{v} \in V$, there exists a probability distribution μ_v over $\boldsymbol{U}$, s.t. for every direction $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{d}}$

$$\langle x, v\rangle^2 \leq \alpha \cdot \mathbb{E}_{u \sim \mu_v}[\langle x, u\rangle^2]$$

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$\boldsymbol{d(v, H_x)}$; $\boldsymbol{d(u, H_x)}$

Geometric interpretation: For any hyperplane $\boldsymbol{H_x}$ (perpendicular to x), the maximum distance of points in V to H_x is "preserved"(in expectation) over U

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equivalently,

$$vv^T \preceq \alpha \cdot \mathbb{E}_{u \sim \mu_v}[uu^T]$$

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⇒

Generalizing the notion of

Combinatorial Spanners

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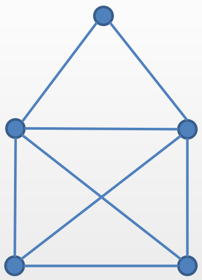
- Let G be a graph with n vertices and m edges.



G

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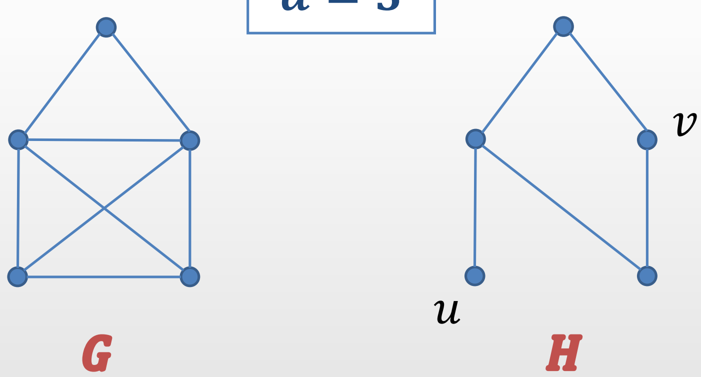
- H is an α spanner if for every edge $f = (u, v) \in E_G$, $d_H(u, v) \leq \alpha$

$\boldsymbol{\alpha = 3}$



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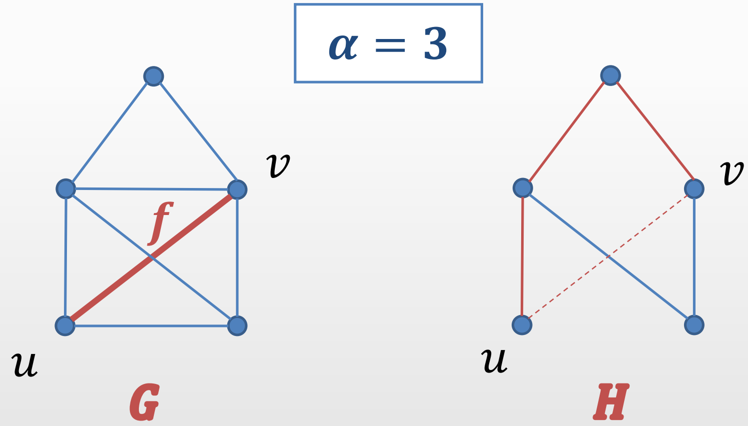

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- Let $V = \{b_f\}$ contain m vectors, $b_f = e_u - e_v$ is n-dimensional

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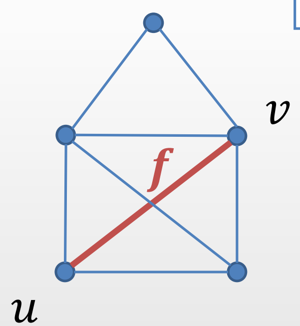
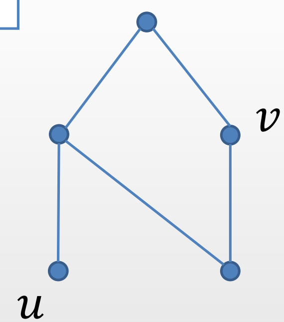


G



H

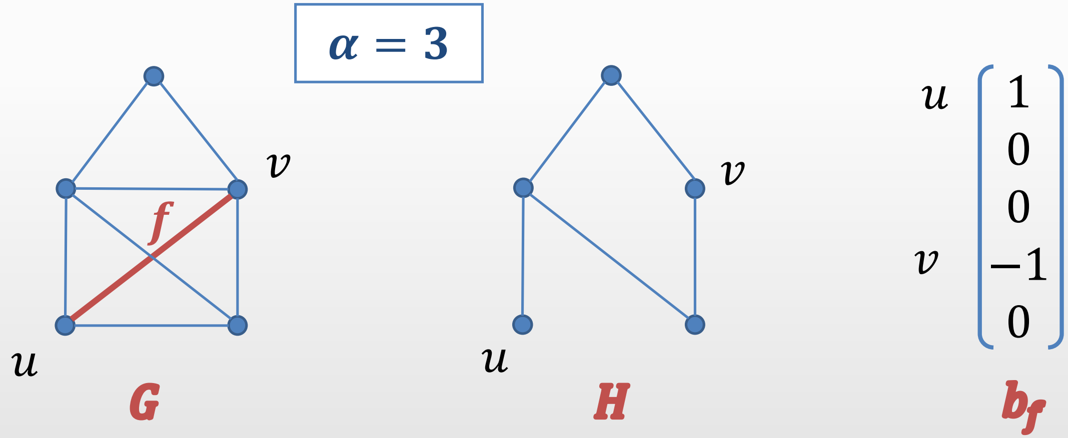
$$\begin{matrix} u \\ \\ \\ v \\ \\ \end{matrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$\boldsymbol{b_f}$

m vectors in n dimensions

Why we call them "spanners"?

- Let G be a graph with n vertices and m edges.
- H is an α spanner if for every edge $f = (u, v) \in E_G$, $d_H(u, v) \leq \alpha$
- Let $V = \{b_f\}$ contain m vectors, $b_f = e_u - e_v$ is n-dimensional



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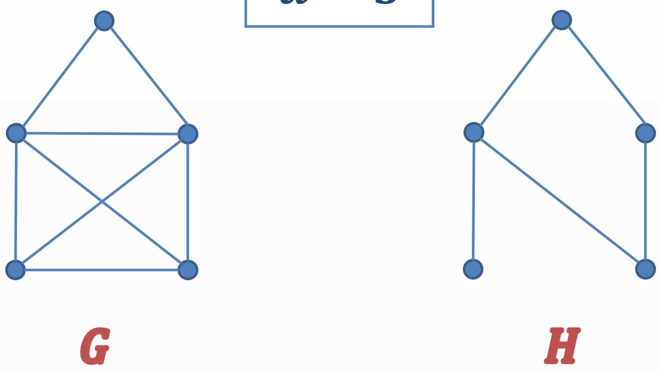
Proof for H being a spectral spanner

$\alpha = 3$



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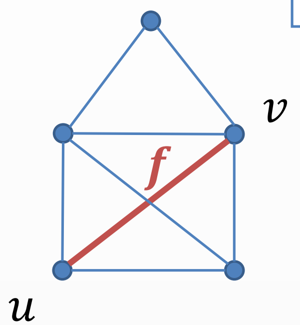

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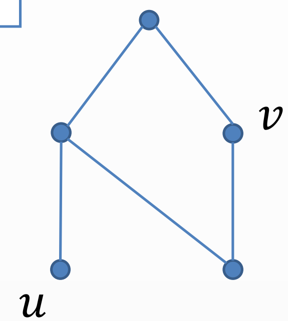

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Take b_f
$f = (u, v)$

Proof for H being a spectral spanner

Define μ_f

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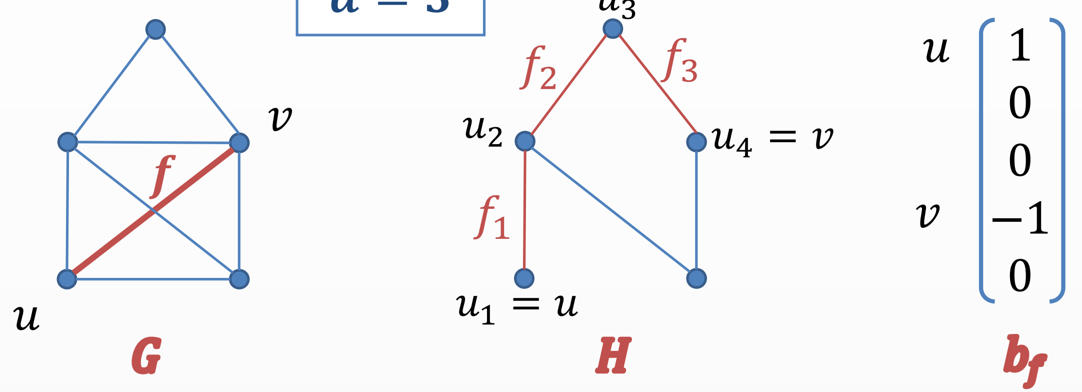

Take b_f
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There exists $f_1, \ldots, f_\alpha$ s.t.

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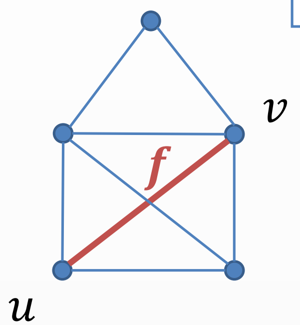

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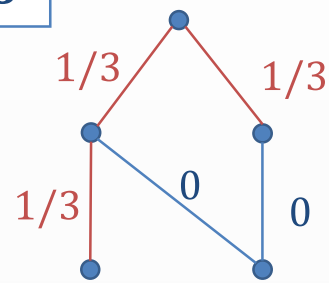

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Goal: $\langle x, b_f\rangle^2 \le \alpha^2 \cdot \mathbb{E}_{e\sim\mu_f}[\langle x, b_e\rangle^2]$

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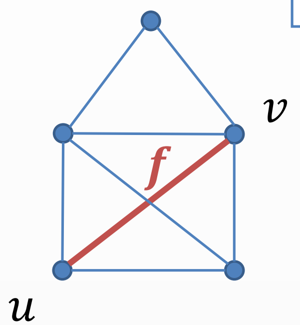

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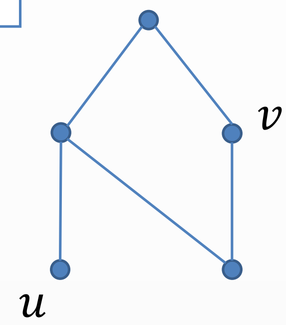

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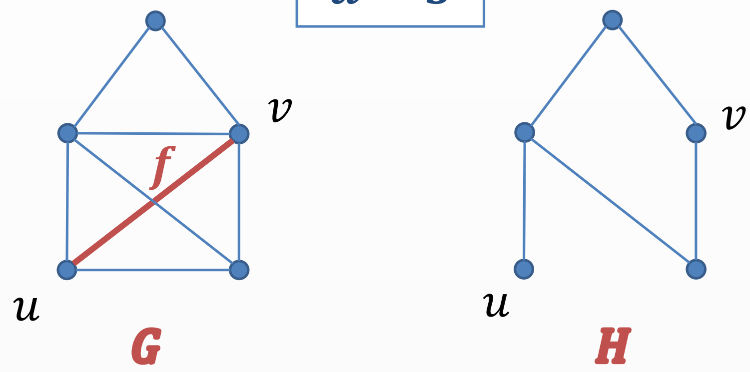

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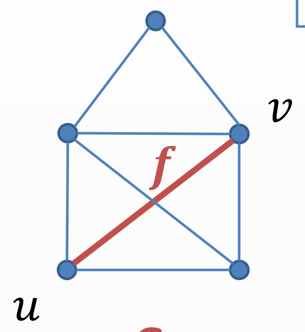

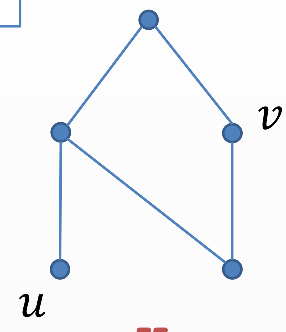

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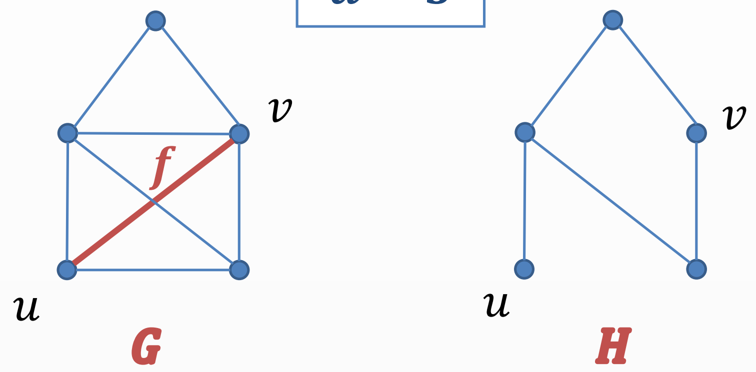

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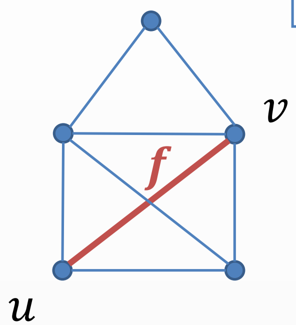

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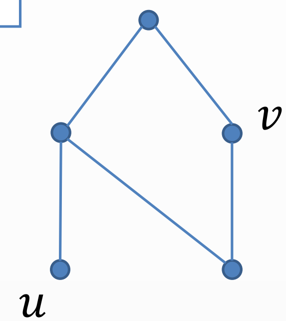

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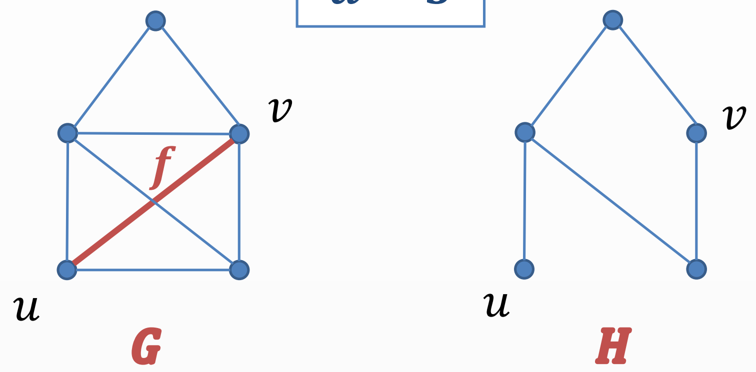

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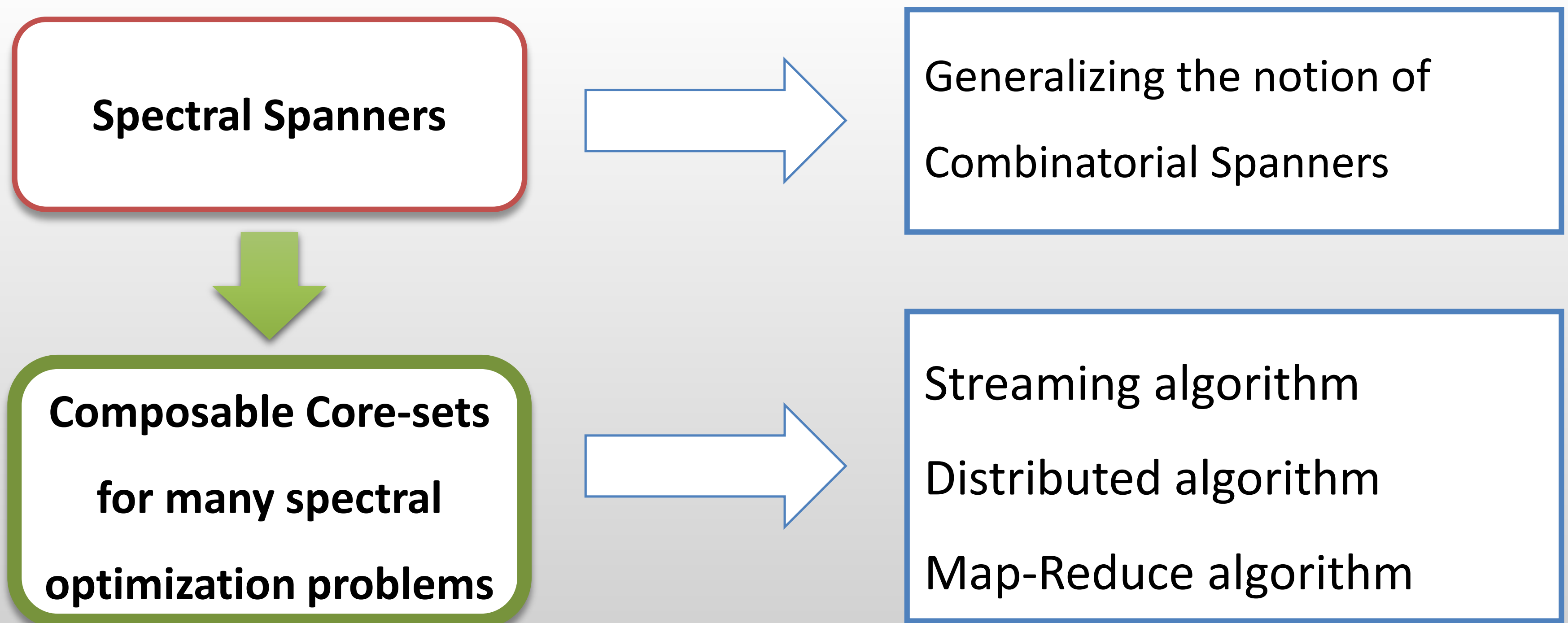

Spectral spanners are good core-sets

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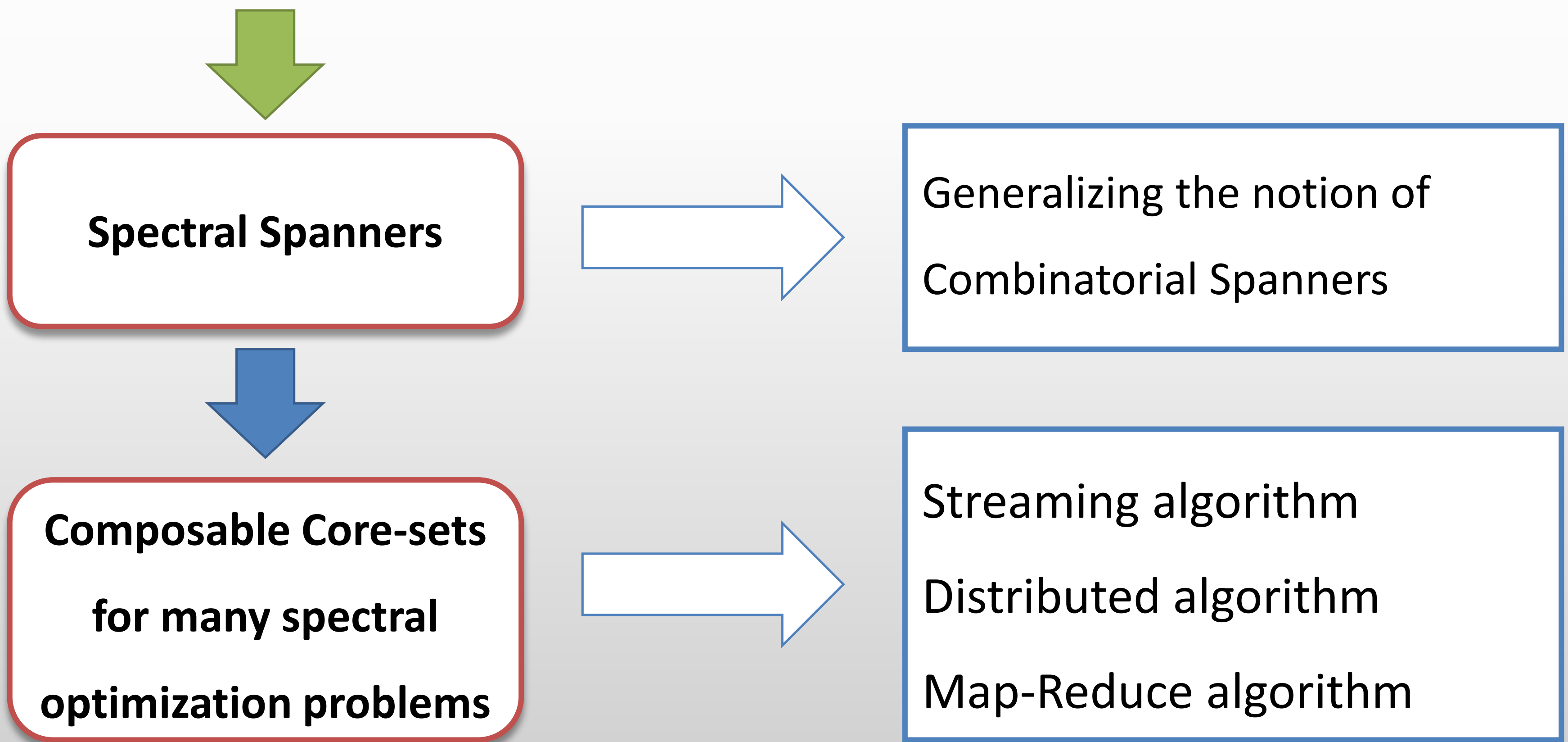
$$\det(OPT) \leq \alpha^d \det(A) \leq (\alpha e)^d \det(T)$$

Will show: spectral spanner with $\alpha = \tilde{O}(d)$ and thus the app. factor is $\tilde{O}(d)^d$.

Overall Picture



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Overall Picture

Weak Spectral Spanners → **Geometric Interpretation:** Preserves maximum distance to any k-dimensional hyperplane.

Spectral Spanners → Generalizing the notion of Combinatorial Spanners

↓

Composable Core-sets for many spectral optimization problems →

- Streaming algorithm
- Distributed algorithm
- Map-Reduce algorithm

Weak Spanners

- Change the order of quantifiers in the spanner:
- **Strong Spanners**: For **every** $\boldsymbol{v}$, there **exists** $\boldsymbol{\mu_v}$, s.t. for **every** $\boldsymbol{x} \in \mathbb{R}^{\boldsymbol{d}}$,

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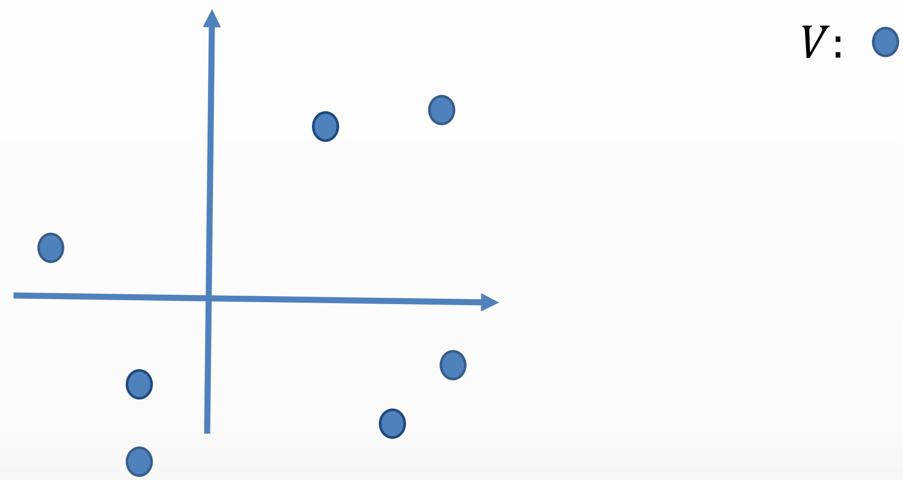

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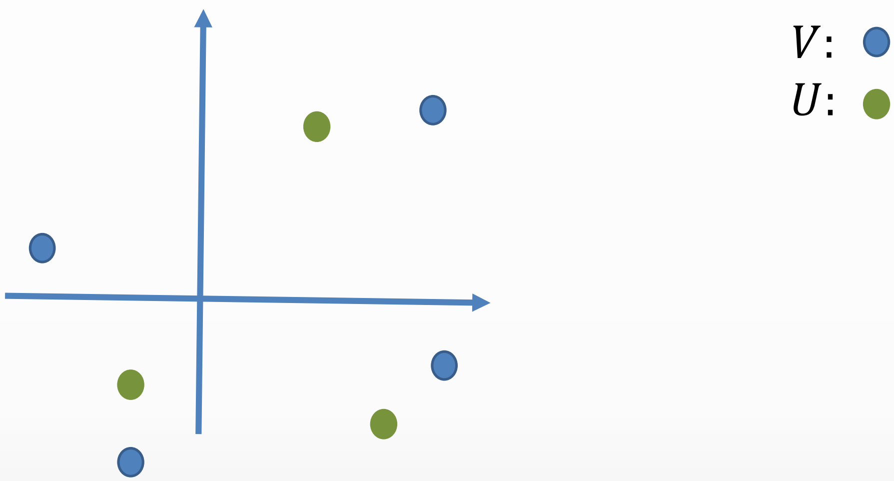

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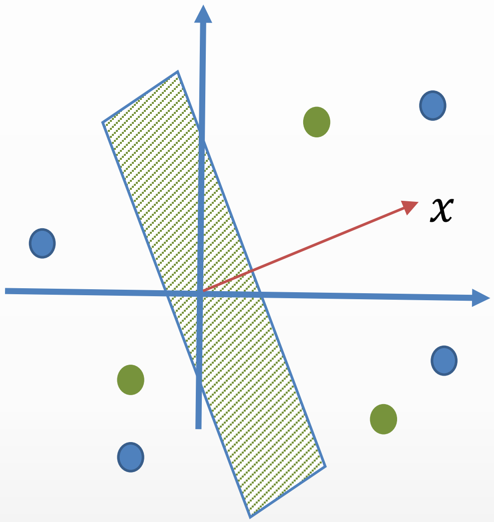

V: (blue dot)
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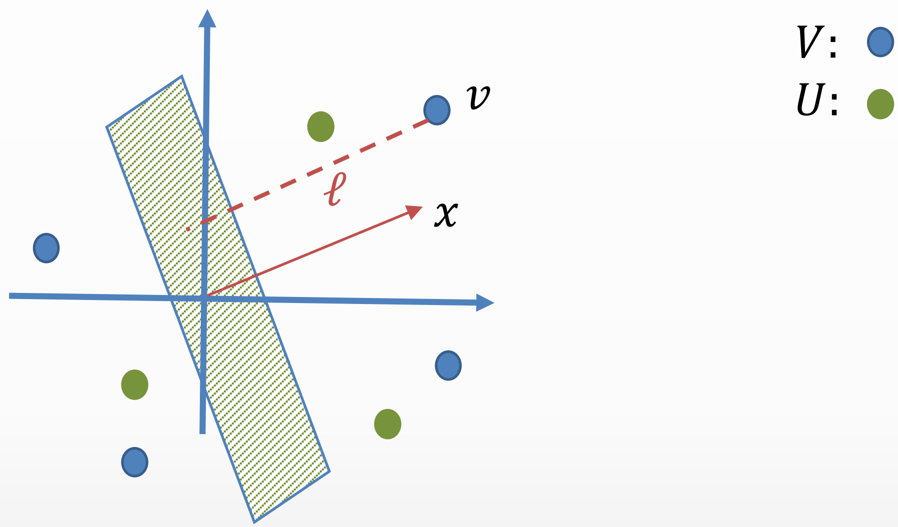

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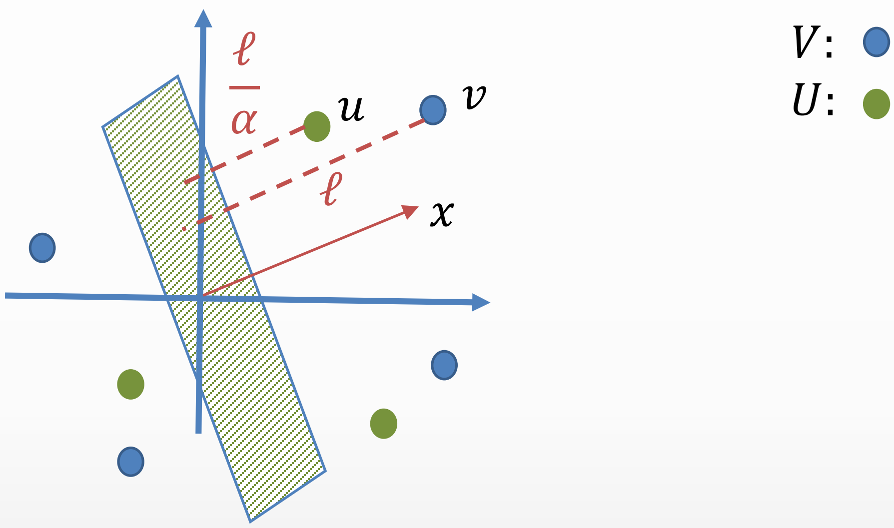

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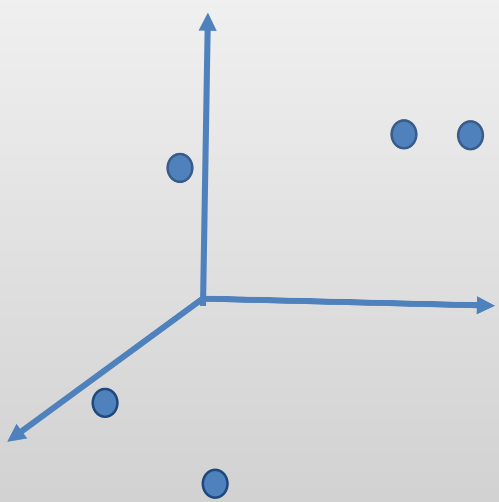

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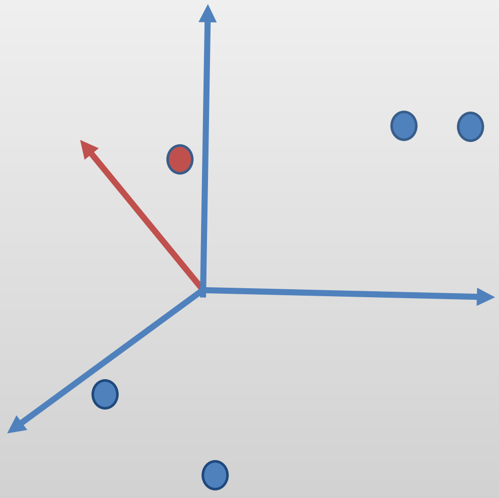

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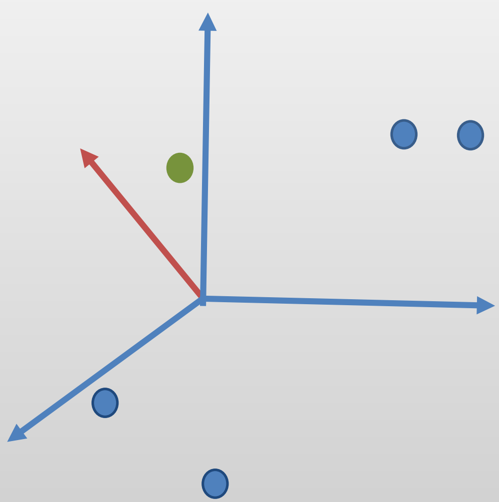

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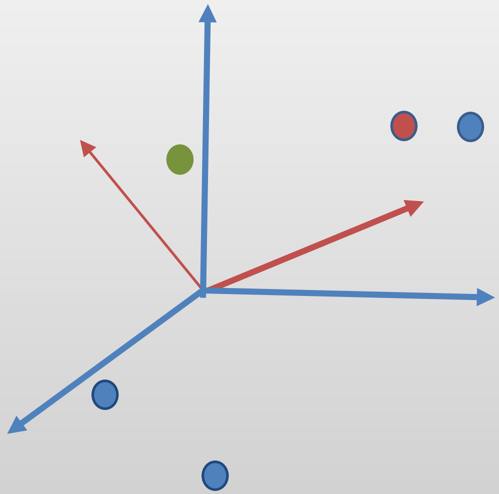

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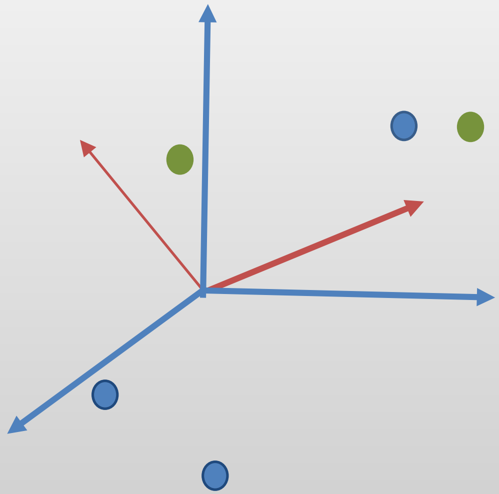

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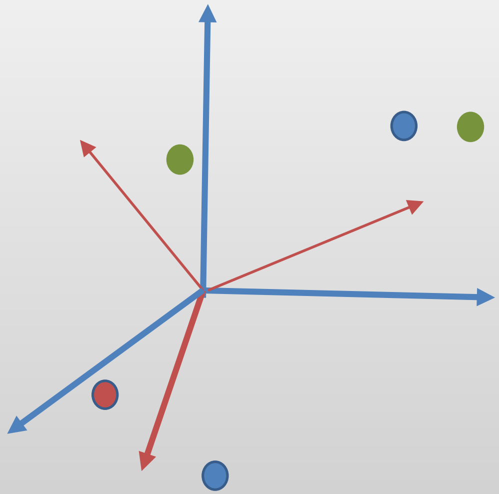

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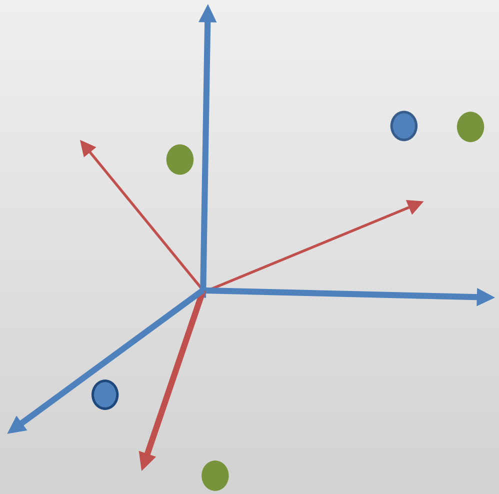

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- **We should bound the size of *U***

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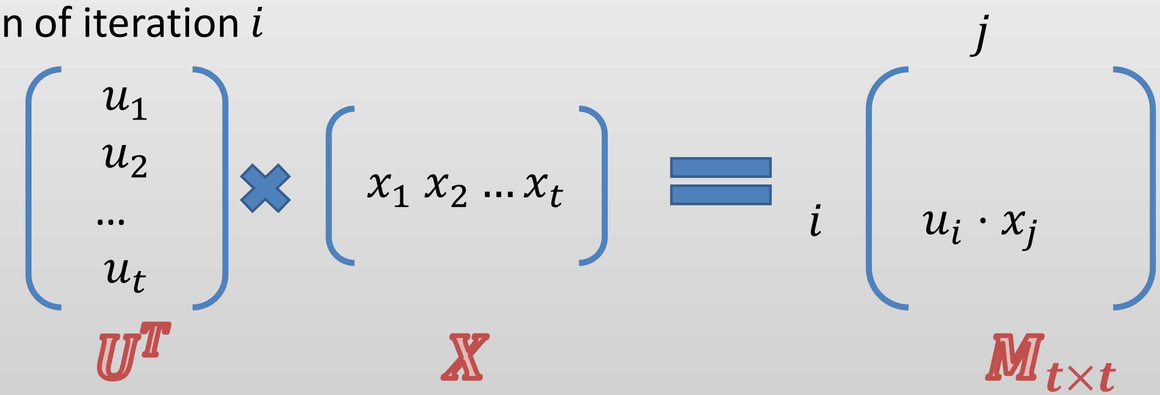
➢ **We should bound the size of U**

- Let $t = |U|$ be the number of iterations
- Let u_i be the vector picked at iteration i in the spanner
- Let x_i be the bad direction of iteration i



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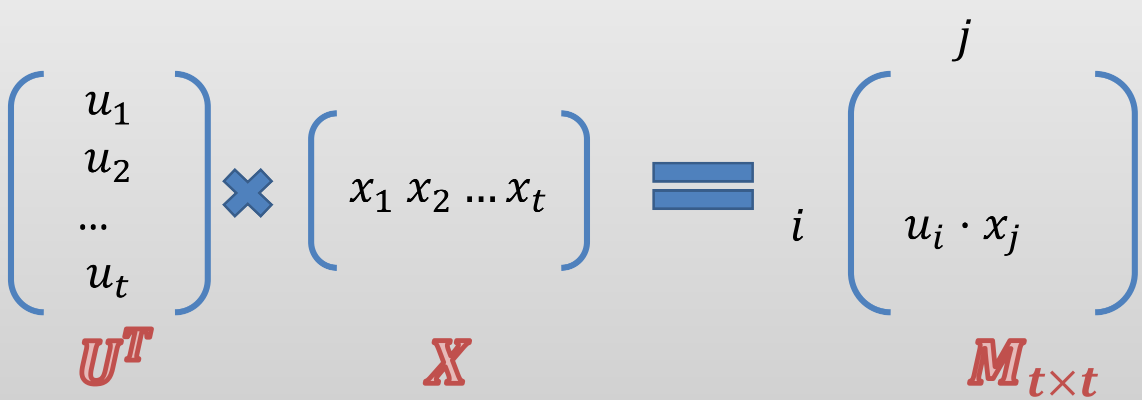
- **Fact:** $rank(M) \leq d$ as M is a product of a $t \times d$ and a $d \times t$ matrix
- **Goal:** Show $rank(M) \geq C \cdot t$
- **Implies:** $t \leq d/C$



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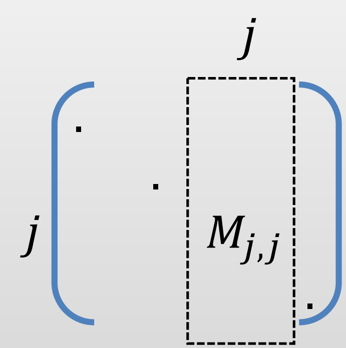

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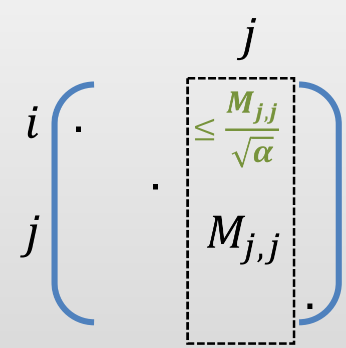

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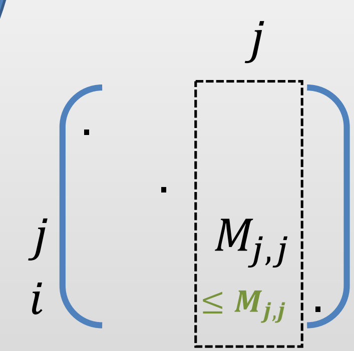

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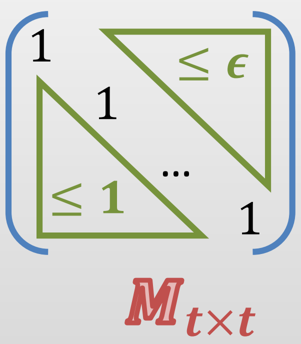
➢ **Divide each column j of M by $M_{j,j}$**

Take the matrix $M_{t\times t}$ where $M_{i,j} = \langle x_i, v_j\rangle / \langle x_j, v_j\rangle$

- All 1 on the diagonal
- For any $i < j$, we have $M_{i,j} \le \left(\frac{1}{\sqrt{\alpha}}\right) := \epsilon$
- For any $i > j$, we have $M_{i,j} \le 1$



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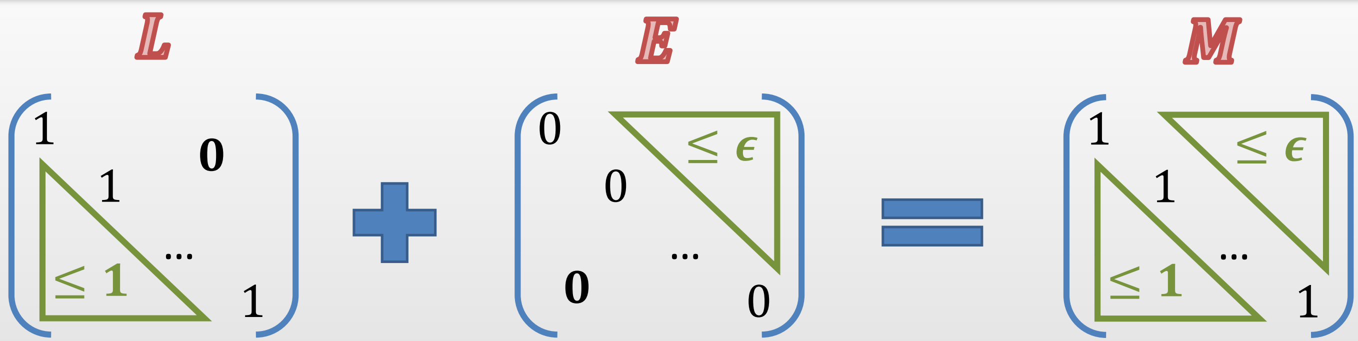

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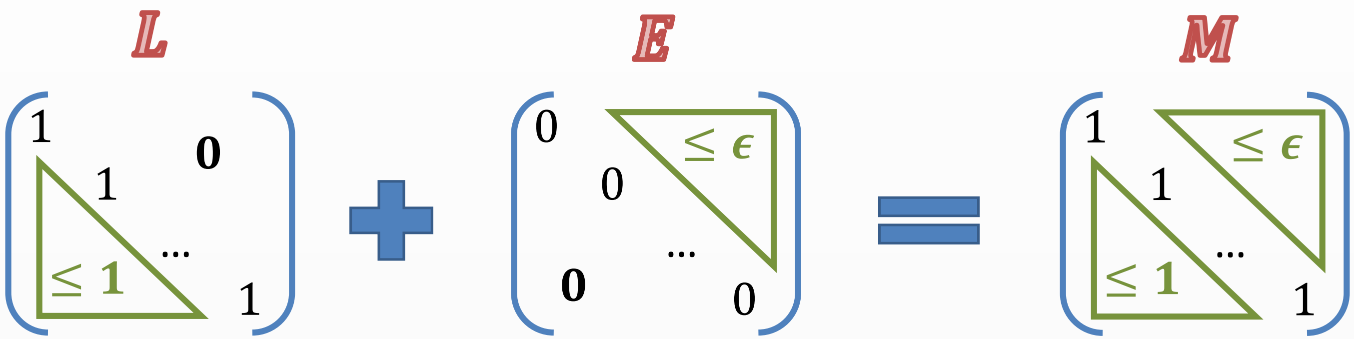

- ***L*** is a diagonally dominant lower triangular
- $rank(L) = t$

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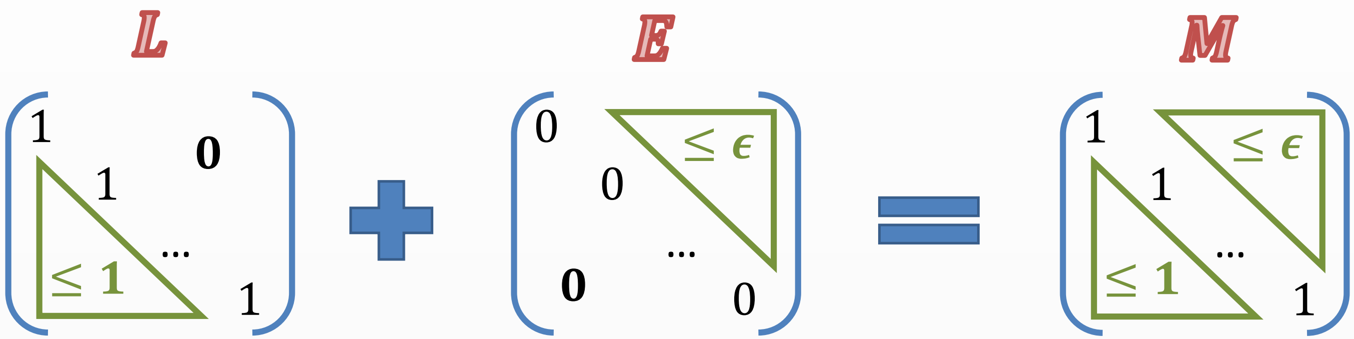

- $\boldsymbol{L}$ is a diagonally dominant lower triangular
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- $\boldsymbol{E}$ has small entries

- **Theorem: Adding $\boldsymbol{E}$ does not decrease the rank "a lot"**

➢**Goal:** Show $rank(M) \geq C \cdot t$

Overall Picture

Weak Spectral Spanners

Spectral Spanners

Composable Core-sets for many spectral optimization problems

Geometric Interpretation:

Preserves maximum distance to any k-dimensional hyperplane.

Generalizing the notion of Combinatorial Spanners

Streaming algorithm

Distributed algorithm

Map-Reduce algorithm

From weak spanners to strong spanners

Theorem: any weak spectral spanner is in fact a strong spectral spanner

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- We need to show existence of μ_v → write an SDP for each v
- Instead consider the dual of SDP
- Use hyperplane separating theorem to show such a distribution exists.

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Generalizing to $\boldsymbol{k < d}$

Spectral Spanners ($\boldsymbol{k = d}$)

A set $\boldsymbol{U} \subseteq \boldsymbol{V}$ is a $\boldsymbol{\alpha}$**-spectral spanner** of V if for every $\boldsymbol{v} \in V$, there exists a probability distribution $\boldsymbol{\mu_v}$ over $\boldsymbol{U}$, s.t.

$$vv^T \preceq \alpha \cdot \mathbb{E}_{u \sim \mu_v}[uu^T]$$

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$$\boldsymbol{A \preceq_k B}$$

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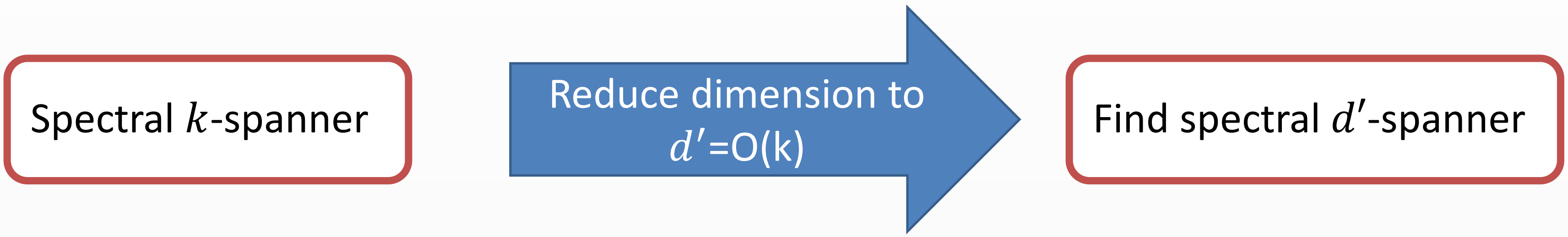
✓ Preserve distances to k-subspaces.

How to find spectral k-spanner?



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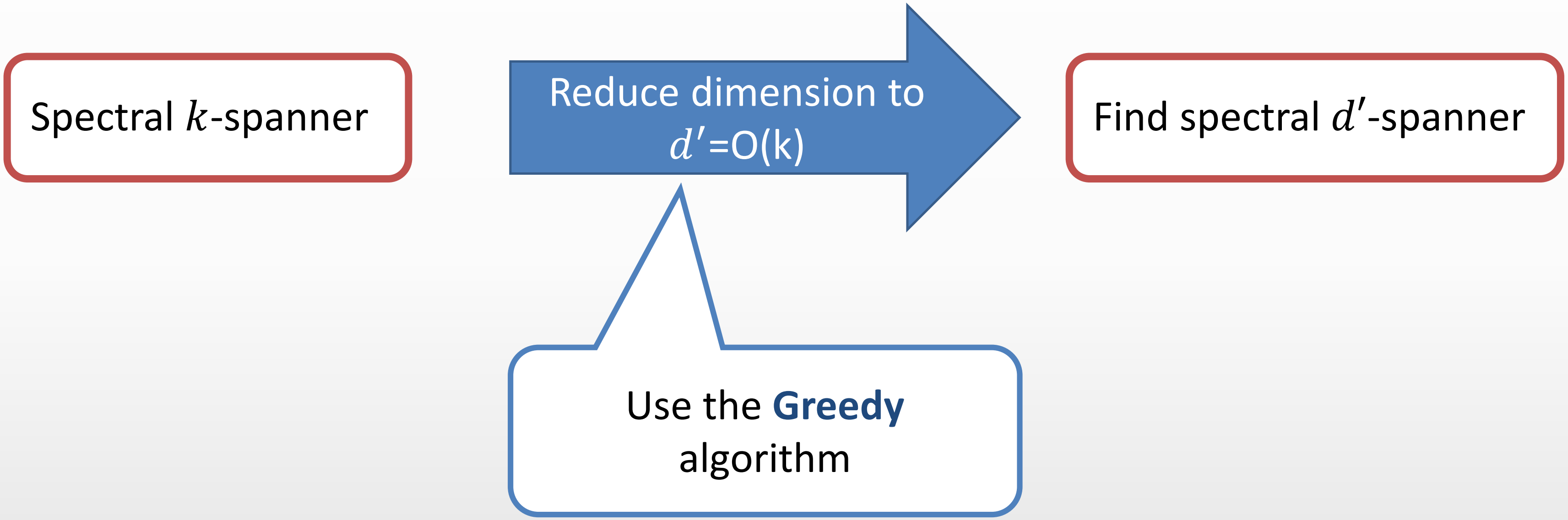
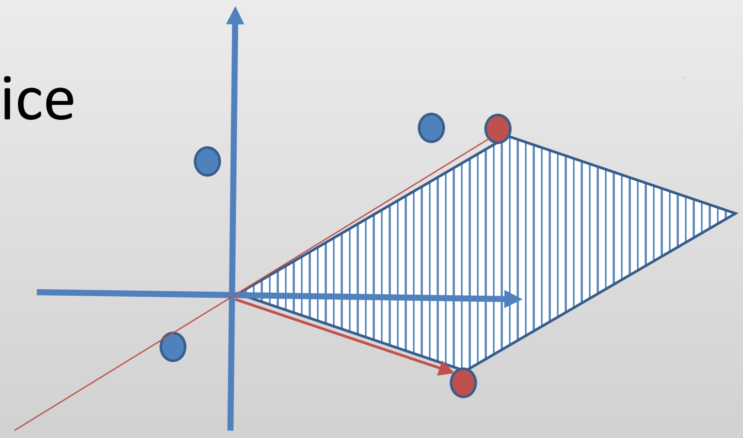


What is known?

- Hard to approximate within a factor of 2^{ck} [CMI'13]
- Best algorithm: e^{k}-approximation [Nik'15]
- **Greedy** is a popular algorithm: achieves approximation factor $k!$
  - $U \leftarrow \emptyset$
  - For k iterations,
    - Add to U the farthest point from the subspace spanned by U
- Greedy performs very well in practice

$\boldsymbol{k = 2}$



Reduction to full dimensional case

Input: a set of points V and a parameter $k < d$

Output: k-spanner of V

- Run The **Greedy** algorithm on V for $2k$ iterations $\rightarrow \boldsymbol{U_1}$
- Let $\boldsymbol{S}$ be the subspace spanned by U_1
- **Project** V on S
- (Now $\boldsymbol{d' = O(k)}$)
- Find d'-spanner on projected $V \rightarrow \boldsymbol{U_2}$
- **Return** $\boldsymbol{U_1 \cup U_2}$

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- Similar results for other spectral optimization problems.

Comparison in practice

- Greedy algorithm
  - Widely used in Practice
  - We showed it achieves $O(C^{k^2})$
- Local Search algorithm
  - Performs better than Greedy but runs ~4 times slower.
  - Achieves $O\left(k^{2k}\right)$
- This algorithm
  - Achieves $\tilde{O}\left(k^k\right)$
  - Performs worse than Local Search and runs slower.

Open Problems

- Other applications of Spectral Spanners?
- Analogue of k-spanners for graphs?
- Composable Core-sets for DPP sampling?

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THANK YOU!